Pure and Mixed Stationary Nash Equilibria for Dynamic Positional Games on Graphs

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Abstract

We consider a special class of positional games with average and discounted payoffs on graphs that generalizes the mean payoff games on graphs. We show that for an arbitrary positional game with average payoffs there exists a Nash equilibrium in mixed stationary strategies and for an arbitrary positional game with discounted payoffs there exists a Nash equilibrium in pure stationary strategies.

Keywords : *Average positional games, discounted positional games, stationary equilibrium.*

1 Introduction and problem formulation

We consider the problem of the existence and determining stationary Nash equilibria for a class of m-player positional games on graphs that generalizes two-player positional games from [1, 3]. The framework of an m-player dynamic positional game is the following:

Let $G = (X, E)$ be a finite oriented graph in which every vertex $x \in X$ has at least one outgoing directed edge $e = (x, y) \in E$. The vertex set X is divided into m disjoint subsets $X_1, X_2, \ldots, X_m$ ($X = X_1 \cup X_2 \cup \ldots \cup X_m$; $X_i \cap X_j = \emptyset, i \neq j$) which represent the position sets of m players. Additionally, on the edge set m functions $c^i : E \to R$, $i = 1, 2, \ldots, m$ are given that assign to each directed edge $e = (x, y) \in E$ the values $c^1_e, c^2_e, \ldots, c^m_e$ that are treated as the rewards for the corresponding players $1, 2, \ldots, m$ if the game passes from a position x to a position y through a directed edge $e = (x, y) \in E$.

On G the following m-person dynamic game is considered. The game starts at a given vertex $x_0 \in X$ at the moment of time $\tau = 0$. The player $i \in \{1, 2, \ldots, m\}$ who is owner of position x_0 makes a move from x_0 to a neighbor vertex $x_1 \in X$ through the directed edge $e_0 = (x_0, x_1) \in E$ and players $1, 2, \ldots, m$ receive the corresponding rewards $c^1_{e_0}, c^2_{e_0}, \ldots, c^m_{e_0}$. Then at the moment of time $\tau = 1$ the player $k \in \{1, 2, \ldots, m\}$ who is owner of position x_1 makes a move from x_1 to a vertex $x_2 \in X$ through the directed edge $e_1 = (x_1, x_2) \in E$ where players $1, 2, \ldots, m$ receive their rewards $c^1_{e_1}, c^2_{e_1}, \ldots, c^m_{e_1}$, and so on, indefinitely. Such a game on G produces the sequence of positions $x_0, x_1, x_2, \ldots x_\tau \ldots$ where x_τ is the position of the game at the moment of time τ. We study this infinite game on graph G for the following two cases:

1) the payoffs of the players represent the average rewards per move, i.e,

$$\omega^i_{v_o} = \lim_{t \to \infty} \inf \frac{1}{t} \sum_{\tau=0}^{t-1} c^i_{e_\tau}, \quad i = 1, 2, \ldots, m;$$

2) the payoffs of the players represents the total discounted rewards with a given discount factor $\lambda, 0 < \lambda < 1$, i.e.

$$\sigma^i_{v_0} = \sum_{\tau=0}^{\infty} \lambda^\tau c^i_{e_\tau}, \quad i = 1, 2, \ldots, m.$$

We call the game in the first case an *average positional game on* G and in the second case we call it *discounted positional game on* G.

The considered games in the case $m = 2$ and $c_e^1 = -c_e^2 = c_e$, $\forall e \in E$ represent, respectively, an *antagonistic average positional game* and an *antagonistic discounted positional game* on G. The zero-sum average positional game of two players on graphs has been studied in [1, 3] where the existence of the value of the game has been proven. Moreover, in these articles it has been shown that players in the game on G can achieve the value by applying the strategies of moves which do not depend on t but depend only on the vertex (position) from which the player is able to move. Therefore, sometimes such strategies are called *positional strategies*; in [3] these strategies are called *stationary strategies*. In fact such strategies can be specified as *pure stationary strategies* because each move through a directed edge in a position of the game is chosen from the set of feasible strategies of moves by the corresponding player with the probability equal to 1 and in each position such a strategy does not change in time. For a non-zero m-player positional game with average payoffs a Nash equilibrium in pure stationary strategies may not exist. This fact has been shown in [3] where an example of a two-player non-zero average positional game that has no Nash equilibrium in pure strategies is constructed. In this contribution we consider the non-zero-sum positional games in mixed stationary strategies. We define a mixed stationary strategy of moves in a position $x \in X_i$ for the player $i \in \{1, 2, \ldots, m\}$ as a probability distribution over the set of feasible moves from x.

2 The main results

The problem of determining Nash equilibria in mixed stationary strategies for positional games on graphs leads to a special class of stochastic games from [4, 5] called *stochastic positional games*. In [5] it is shown that such a class of games possesses Nash equilibria in mixed stationary strategies. In general, for an average stochastic game with finite state and action spaces a stationary Nash equilibrium may not exist (see [2]). Therefore we shall use the stochastic positional games for studying the problem of the existence and determining stationary Nash equilibria for non-zero-sum positional games on graphs with average and discounted payoffs for the players. An m-player stochastic positional game consists of the following elements: A finite set of states X; a given partition $X = X_1 \cup X_2 \cup \ldots \cup X_m$ of X; where X_i represents the position set of player $i \in \{1, 2, \ldots, m\}$; a finite set of actions $A(x)$ in each state $x \in X$; a step reward $f^i(x, a)$ for each player $i \in \{1, 2, \ldots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$; a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \to [0, 1]$ that gives the transition probabilities $p_{x,y}^a$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum_{y \in X} p_{x,y}^a = 1, \forall x \in X, a \in A(x)$ and the starting state $x_0 \in X$.

The game starts in the state x_0 at the moment of time $\tau = 0$. The player $i \in \{1, 2, \ldots, m\}$ who is the owner of the state position x_0 ($x_0 \in X_i$) chooses an action $a_0 \in A(x_0)$ and determines the rewards $f^1(x_0, a_0), f^2(x_0, a_0), \ldots, f^m(x_0, a_0)$ for the corresponding players $1, 2, \ldots, m$. After that the game passes to a state $y = x_1 \in X$ according to the probability distribution $\{p_{x_0,y}^{a_0}\}$. At the moment of time $\tau = 1$ the player $k \in \{1, 2, \ldots, m\}$ who is the owner of the state position x_1 ($x_1 \in X_k$) chooses an action $a_1 \in A(x_1)$ and players $1, 2, \ldots, m$ receive the corresponding rewards $f^1(x_1, a_1), f^2(x_1, a_1), \ldots, f^m(x_1, a_1)$. Then the game passes to a state $y = x_2 \in X$ according to the probability distribution $\{p_{x_1,y}^{a_1}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \ldots, x_\tau, a_\tau, \ldots$ that defines a stream of stage rewards $f^1(x_\tau, a_\tau), f^2(x_\tau, a_\tau), \ldots, f^m(x_\tau, a_\tau)$, $\quad \tau = 0, 1, 2, \ldots$.

The average stochastic positional game is the game with payoffs of the players

$$\omega_{x_0}^i = \liminf_{t \to \infty} \mathsf{E}\Big(\frac{1}{t} \sum_{\tau=0}^{t-1} f^i(x_\tau, a_\tau)\Big), \quad i = 1, 2, \ldots, m$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players in their position sets and given starting state x_0.

The discounted stochastic positional game is the game with payoffs of the players

$$\sigma_{x_0}^i = \mathsf{E}\Big(\sum_{\tau=0}^{t-1} \lambda^\tau f^i(x_\tau, a_\tau)\Big), \quad i = 1, 2, \dots, m,$$

where λ is a given discount factor, $0 < \lambda < 1$.

In [4, 5, 6] it has been shown that the considered games possesses Nash equilibria in stationary strategies. A *strategy of player* $i \in \{1, 2, \dots, m\}$ in a stochastic positional game is a mapping s^i that gives for every state $x_t \in X_i$ a probability distribution on $A(x_t)$. If these probabilities take only 0 and 1, then s^i is called *pure strategy*, otherwise it is called *mixed strategy.* If these probabilities depend only on the state $x_t = x \in X_i$ (i. e. s^i does not depend on t), then s^i is called *stationary strategy*, otherwise s^i is called *non-stationary strategy.* Thus, for the set of mixed stationary strategies $\mathbf{S}^i$ of player i we can identify with the set of solutions of the system

$$\begin{cases} \sum\limits_{a \in A(x)} s_{x,a}^i = 1, & \forall x \in X_i; \\ s_{x,a}^i \geq 0, & \forall x \in X_i, \ \ \forall a \in A(x). \end{cases} \tag{1}$$

Each basic solution s^i of this system represents a pure stationary strategy of player $i \in \{1, 2, \dots, m\}$ while a non basic solution corresponds to a mixed stationary strategy.

Let $\mathbf{s}= (s^1, s^2, \dots, s^m) \in \mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \dots \times \mathbf{S}^m$ be a profile of stationary strategies of the game on G. Then the probability transition matrix $P^{\mathbf{s}} = (p_{x,y}^{\mathbf{s}})$ of the Markov process induced by $\mathbf{s}$ can be calculated as follows: $p_{x,y}^{\mathbf{s}} = \sum\limits_{a \in A(x)} s_{x,a}^i p_{x,y}^a$ for $x \in X_i$, $i = 1, 2, \dots, m$. If we denote by $Q^{\mathbf{s}} = (q_{x,y}^{\mathbf{s}})$ the limiting probability matrix of matrix $P^{\mathbf{s}}$ then the average rewards per transition $\omega_{x_0}^1(\mathbf{s})$, $\omega_{x_0}^2(\mathbf{s}), \dots, \omega_{x_0}^m(\mathbf{s})$ for the players induced by profile $\mathbf{s}$ are determined as follows $\omega_{x_0}^i(\mathbf{s}) = \sum_{k=1}^m \sum_{y \in X_k} q_{x_0,y}^{\mathbf{s}} f^i(y, s^k)$, $\quad i = 1, 2, \dots, m$, where $f^i(y, s^k) = \sum\limits_{a \in A(y)} s_{y,a}^k f^i(y, a)$, for $y \in X_k$, $k \in \{1, 2, \dots, m\}$ represents the average reward of player i in the state $y \in X_k$ when player k uses the strategy s^k. The functions $\omega_{x_0}^1(\mathbf{s})$, $\omega_{x_0}^2(\mathbf{s})$, $\dots$, $\omega_{x_0}^m(\mathbf{s})$ on $\mathbf{S}$ in such a way determine a game in normal form that we denote by $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\omega_{x_0}^i(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *average stochastic positional game in mixed stationary strategies* on graph G. In the extended form this game is determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x \in X}, \{f^i(x, a)\}_{i=\overline{1,m}}, p, x_0)$. In such a way the functions $\omega_{x_0}^1(\mathbf{s}), \omega_{x_0}^2(\mathbf{s}), \dots, \omega_{x_0}^m(\mathbf{s})$ on S, determine the game $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega_{x_0}^i(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that corresponds to the *average stochastic positional game in pure strategies.*

An average stochastic positional games can be considered also when the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. So, in this case the game starts in the state $x \in X$ with probability $\theta_x > 0$ where $\sum\limits_{x \in X} \theta_x = 1$. For such a game the payoff functions on $\mathbf{S}$ are defined as follows $\psi_\theta^i(\mathbf{s}) = \sum\limits_{x \in X} \theta_x \omega_x^i(\mathbf{s})$, $\quad i = 1, 2, \dots, m$ and we obtain the game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi_\theta^i(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that is determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x \in X}, \{f^i(x, a)\}_{i=\overline{1,m}}, p, \{\theta_x\}_{x \in X})$. In the case $\theta_x = 0, \forall x \in X \setminus \{x_0\}$, $\theta_{x_o} = 1$ this game becomes a stochastic positional game with a fixed starting state $x_0 \in X$.

Similarly, a discounted stochastic positional game can be considered in the case when the game starts in the state that is chosen randomly according to a given distribution $\{\theta_x\}$ on X. In this case the payoffs of the players are defined as $\sigma_\theta^i(\mathbf{s}) = \sum\limits_{x \in X} \theta_x \sigma_x^i(\mathbf{s})$, $\quad i = 1, 2, \dots, m$.

In [5] it has been shown that $\mathbf{S}^i$ and the payoffs $\psi_\theta^i(\mathbf{s})$, $\quad i = 1, 2, \dots, m$ for the average stochastic positional game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \ \{\psi_\theta^i(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ can be specified as follows: The set of stationary strategies $\mathbf{S}^i$, $i \in \{1, 2, \dots m\}$ represent the set of solutions of system (1) and the payoffs on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \dots \times \mathbf{S}^m$ are defined as

$$\psi_\theta^i(s^1, s^2, \dots, s^m) = \sum_{k=1}^m \sum_{x \in X_k} \sum_{a \in A(x)} s_{x,a}^k f^i(x, a) q_x, \qquad i = 1, 2, \dots, m, \tag{2}$$

where q_x for $x \in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y - \sum\limits_{k=1}^{m} \sum\limits_{x\in X_k} \sum\limits_{a\in A(x)} s^k_{x,a}\ p^a_{x,y}\ q_x = 0, & \forall y \in X; \\ \\ q_y + w_y - \sum\limits_{k=1}^{m} \sum\limits_{x\in X_k} \sum\limits_{a\in A(x)} s^k_{x,a}\ p^a_{x,y}\ w_x = \theta_y, & \forall y \in X \end{cases} \tag{3}$$

for an arbitrary fixed $\mathbf{s} = (s^1, s^2, \ldots, s^m) \in \mathbf{S}$

Theorem 1 *If $\theta_y > 0, \forall y \in X$ then the game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ has a Nash equilibrium $\mathbf{s}^*= (s^{1^*}, s^{2^*}, \ldots, s^{m*}) \in \mathbf{S}$ which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{f^i(x,a)\}_{i=\overline{1,m}}, p,\ \{\theta_y\}_{y\in X})$. Moreover, $\mathbf{s}^* = (s^{1^*}, s^{2^*}, \ldots, s^{m*})$ is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\omega^i_y(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ with an arbitrary starting state $y \in X$.*

Based on this result we can conclude that an average positional game on graph G also has a Nash equilibrium in stationary strategy because in a position $x \in X_i$ of the game an outgoing directed edges $e = (x, y)$ correspond to an action $a = (x, y)$ that provide a move from x to y with probability $p^a_{x,y} = 1$, i.e. the static game model (1)-(3) can be applied. For the positional game on G with discounted payoffs there exists a stationary Nash equilibria because there exist stationary Nash equilibria for a discounted stochastic game in general. In [6] it has been shown that for the discounted positional game on a graph there exists stationary equilibria in pure strategies.

3 Conclusion and perspectives

The presented results show that positional games on graphs with average and discounted payoffs possess Nash equilibria in stationary strategies. Additionally, for a discounted positional game on graphs there exist stationary Nash equilibria in pure strategies. The obtained results may be useful for the elaboration of the algorithms for determining the optimal strategies of the players in such games.

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